

A generalization of Montgomery-Yang correspondence

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Abstract

In this paper, we want to construct a one-to-one correspondence from the set of diffeomorphism classes of spin d -twisted homology $\mathbb{C}P^3$ to the set of isotopy classes of the embedding from S^3 to S^6 , which is a generalization of the Montgomery-Yang correspondence. Furthermore, we will apply this generalized correspondence to prove the existence of free involution on these d -twisted homology $\mathbb{C}P^3$.

1 Introduction

1.1 Montgomery-Yang correspondence

Let Π be the set of diffeomorphism classes of homotopy $\mathbb{C}P^3$ and $\Sigma^{6,3}$ be the set of isotopy classes of the embedding from S^3 to S^6 . By [1] and [2], $\Sigma^{6,3}$ admits an abelian group structure with $\Sigma^{6,3} \cong \mathbb{Z}$. In their paper [9], Montgomery and Yang constructed a one-to-one correspondence from Π to $\Sigma^{6,3}$:

$$\Phi : \Pi \longrightarrow \Sigma^{6,3}.$$

Furthermore, they constructed an operation $*$ on Π to make $(\Pi, *)$ admit an abelian group structure with unit $\mathbb{C}P^3$ and proved that this correspondence $\Phi : \Pi \longrightarrow \Sigma^{6,3} \cong \mathbb{Z}$ is also a group isomorphism.

By the classification theorems of closed 1-connected 6-manifold with torsion free integer homology group (cf [10]), we may quickly know Π , as a set, is one-to-one correspondence to \mathbb{Z} . But we can not obtain the group structure of Π from those classification theorems, which is one of the key points of the paper of Montgomery and Yang [9].

1.2 Main results

In this paper, we mainly concern with the set Π_d^6 of the diffeomorphism classes of spin d -twisted homology $\mathbb{C}P^3$, which is a class of closed 1-connected smooth spin 6-manifolds with cohomology ring $\mathbb{Z}[x, y]/(x^2 - dy, y^2)$, $\deg x = 2$, $\deg y = 4$, $d \in \mathbb{Z}$.

Follow the idea of [9], we first construct an operation \natural on Π_d^6 to make (Π_d^6, \natural) become an abelian group. Second, we will construct a group isomorphism:

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Theorem 1.1. $\Phi : \Pi_d^6 \cong \Sigma^{6,3}$.

As an application, we will prove an existence theorem on Π_d^6 :

Theorem 1.2. *For any $M \in \Pi_d^6$, there exists a free involution on M .*

For the involution questions on homotopy projective 3-spaces and d -twisted homology $\mathbb{C}P^3$, we refer to [5], [6].

The organization of this paper is as follows. In section 2, we will discuss the basic properties of the element in Π_6^d . In section 3, the group structure of Π_6^d will be concerned and Theorem 1.1 will be proved. Finally, we will prove theorem 1.2 in section 4.

1.3 Notations

$$D^n := \{x \in \mathbb{R}^n \mid |x| \leq 1\}.$$

$$D_+^n := \{(x_1, \dots, x_n) \in D^n \mid x_n \geq 0\}.$$

$$D_-^n := \{(x_1, \dots, x_n) \in D^n \mid x_n \leq 0\}.$$

$$E^n := (D^n)^\circ = \{x \in \mathbb{R}^n \mid |x| < 1\}.$$

$$S^{n-1} := \{x \in \mathbb{R}^n \mid |x| = 1\}.$$

2 d -twisted homology $\mathbb{C}P^3$

2.1 Definition of d -twisted homology $\mathbb{C}P^3$

Definition 2.1. *Let M be a closed smooth 1-connected 6-manifold. We call M a d -twisted homology $\mathbb{C}P^3$ if the cohomology ring $H^*(M, \mathbb{Z})$ of M is isomorphic to:*

$$\mathbb{Z}[x, y]/(x^2 - dy, y^2), \quad d \in \mathbb{Z}, \quad \deg x = 2, \quad \deg y = 4.$$

Since $\mathbb{Z}[x, y]/(x^2 - dy, y^2) \cong \mathbb{Z}[x, y]/(x^2 + dy, y^2)$, we can always assume $d \geq 0$.

Example 2.2. *The first example is 3-dimensional complex projective space $\mathbb{C}P^3$. It is simply connected and its cohomology ring is isomorphic to $\mathbb{Z}[x]/(x^4)$, $\deg x = 2$. By definition, $\mathbb{C}P^3$ is 1-twisted. Furthermore, any smooth closed manifold M which is homotopic equivalent to $\mathbb{C}P^3$ is also a 1-twisted homology $\mathbb{C}P^3$.*

Example 2.3. $S^2 \times S^4$ is a 0-twisted homology $\mathbb{C}P^3$.

Example 2.4. *Let F_d be a smooth hypersurface in $\mathbb{C}P^4$ of degree $d > 0$, for example, the Fermat hypersurface: $\{[x_0, x_1, x_2, x_3, x_4] \in \mathbb{C}P^4 \mid \sum_{i=0}^4 x_i^d = 0\}$. By Lefschetz's hyperplane section theorem (c.f. [7]), the pair $(\mathbb{C}P^4, F_d)$ is 3-connected and F_d admits a connected sum decomposition (cf [10]):*

$$F_d \cong M_d \# \frac{b_3(F_d)}{2} S^3 \times S^3.$$

By calculation, $H^*(M_d, \mathbb{Z}) \cong \mathbb{Z}[x, y]/(x^2 - dy, y^2)$, $\deg x = 2, \deg y = 4$. So for any $d > 0$, there always exists a d -twisted homology $\mathbb{C}P^3$.

Remark 2.5. In [4], Libgober and Wood called a simply connected CW complex X of dimension 6 a d -twisted homology $\mathbb{C}P^3$ if $H^*(X, \mathbb{Z}) \cong \mathbb{Z}[x, y]/(x^2 - dy, y^2)$, $d \in \mathbb{Z}$, $\deg x = 2$, $\deg y = 4$. In our paper, we only concern with closed smooth 6-manifolds, so we make a little change of their definitions. In some references, these manifolds are also called fake projective 3-spaces.

2.2 A geometric construction of d twisted homology $\mathbb{C}P^3$

Let $j_2 : S^3 \hookrightarrow S^2 \times S^3$ be the standard embedding. Let $p_1 : S^2 \times S^3 \rightarrow S^2$ and $p_2 : S^2 \times S^3 \rightarrow S^3$ be the projection to the first and second factor of $S^2 \times S^3$. For an orientation reversing diffeomorphism $h : S^2 \times S^3 \rightarrow S^2 \times S^3$ with

- (1). The map $p_2 \circ h \circ j_2 : S^3 \hookrightarrow S^2 \times S^3 \rightarrow S^2 \times S^3 \rightarrow S^3$ has degree -1 .
- (2). The Hopf invariant of $p_1 \circ h \circ j_2 : S^3 \hookrightarrow S^2 \times S^3 \rightarrow S^2 \times S^3 \rightarrow S^3$ is d .

Then we can construct a closed 6-manifold $M(h)$ by gluing two copies of $S^2 \times D^4$ along their boundaries through this orientation reversing diffeomorphism h :

$$M(h) := (S^2 \times D^4) \cup_h (S^2 \times D^4).$$

Lemma 2.6. $M(h)$ is a d -twisted homology $\mathbb{C}P^3$.

Proof. By Van-Kampen's theorem, $M(h)$ is simply connected. The Meyer-Vietoris exact sequence of $M(h)$ shows that the group structure of $H^*(M(h), \mathbb{Z}) = \bigoplus_{i=0}^6 H^i(M(h), \mathbb{Z})$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}z$, with $\deg x = 2$, $\deg y = 4$, and $\deg z = 6$. By Poincaré duality, we can choose $x \cup y$ as a generator of $\mathbb{Z}z$ and $H^*(M(h), \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}(x \cup y)$.

Next we need to find the relation between $x \cup x$ and y , which also determines the ring structure of $H^*(M(h), \mathbb{Z})$. Since $S^2 = E^2 \cup pt$,

$$M(h) = (S^2 \times D^4) \cup_h (S^2 \times D^4) = (E^2 \times E^4) \cup ((pt \times D^4) \cup_{hj_2} (S^2 \times D^4)).$$

Therefore,

$$M(h) - E^6 \cong M(h) - E^2 \times E^4 \cong (pt \times D^4) \cup_{hj_2} (S^2 \times D^4) \simeq D^4 \cup_{p_1 h j_2} S^2$$

and we get the ring isomorphism $H^*(M(h) - E^6, \mathbb{Z}) \cong H^*(D^4 \cup_{p_1 h j_2} S^2, \mathbb{Z})$. By the definition of Hopf invariant, the cohomology ring $H^*(D^4 \cup_{p_1 h j_2} S^2, \mathbb{Z}) \cong \mathbb{Z}[x, y]/(x^2 - dy, xy, y^2)$ with $\deg x = 2$, $\deg y = 4$, where d is the Hopf invariant of the map $p_1 h j_2$.

On the other hand, the restriction map $H^m(M(h), \mathbb{Z}) \rightarrow H^m(M(h) - E^6, \mathbb{Z})$ is isomorphic when $m \neq 6$ and we get the relation $x \cup x = dy$ in $H^*(M(h), \mathbb{Z})$. Finally, combine the group structure of $H^*(M(h), \mathbb{Z})$ and the relation $x \cup x = dy$, we see the cohomology ring $H^*(M(h), \mathbb{Z}) \cong \mathbb{Z}[x, y]/(x^2 - dy, y^2)$, $\deg x = 2$, $\deg y = 4$. \square

What about the converse? First, following [9], let M be a d -twisted homology $\mathbb{C}P^3$ and $i : S^2 \hookrightarrow M$ be an embedding, we call $i : S^2 \hookrightarrow M$ a **primary embedding** if $i_* : H_2(S^2, \mathbb{Z}) \rightarrow H_2(M, \mathbb{Z})$ maps the generator $[S^2] \in H_2(S^2, \mathbb{Z})$ to the generator of $H_2(M, \mathbb{Z})$ which represents the orientation of M .

Example 2.7. For $M(h) = (S^2 \times D^4) \cup_h (S^2 \times D^4)$ and each copy $S^2 \times D^4$, the inclusion $S^2 \hookrightarrow S^2 \times D^4 \subset M(h)$ induces isomorphism $H_2(S^2, \mathbb{Z}) \cong H_2(S^2 \times D^4, \mathbb{Z}) \cong H_2(M(h), \mathbb{Z})$. Since the attaching map h is orientation-reversing, we can make these two copies $S^2 \times D^4 \subset M(h)$ preserve the orientation. Then these two inclusions $S^2 \hookrightarrow S^2 \times D^4 \subset M(h)$ are primary embeddings.

Lemma 2.8. Let M be a d -twisted homology \mathbb{CP}^3 and $i : S^2 \rightarrow M$ be a primary embedding. The normal bundle η_i of i is trivial if and only if M is spin, i.e. $w_2(M) = 0$.

Proof. The normal bundle η_i is trivial if and only if the second obstruction $\sigma_2 \in H^2(S^2, \pi_1(SO(4)/SO(1))) = H^2(S^2, \pi_1(SO(4))) = H^2(S^2, \mathbb{Z}/2\mathbb{Z})$ is zero. From [8], this obstruction σ_2 is equal to the second Stiefel-Whitney class $w_2(\eta_i)$, which is also equal to $i^*w_2(M)$.

Since i is a primary embedding and $\pi_1(M) = 1$, the restriction map $i^* : H^2(M, \mathbb{Z}/2\mathbb{Z}) \rightarrow H^2(S^2, \mathbb{Z}/2\mathbb{Z})$ is an isomorphism. So η_i is trivial if and only if $w_2(M) = 0$. \square

Corollary 2.9. $M(h)$ is spin.

Proof. $M(h) = (S^2 \times D^4) \cup_h (S^2 \times D^4)$, choose one primary embedding $S^2 \hookrightarrow S^2 \times D^4 \subset M(h)$. The normal bundle of S^2 is trivial. \square

Proposition 2.10. Let M be a d -twisted homology \mathbb{CP}^3 , then M is spin if and only if

$$M \cong (S^2 \times D^4) \cup_h (S^2 \times D^4),$$

where $h : S^2 \times S^3 \rightarrow S^2 \times S^3$ is an orientation reversing diffeomorphism with (1). $p_2 \circ h \circ j_2$ has degree -1 , (2). The Hopf invariant of $p_1 \circ h \circ j_2$ is d .

Proof. Only one direction needs to be proved. Assume M is spin. According to Whitney's embedding theorem, we can represent the generator of $H_2(M, \mathbb{Z})$ which represents the orientation class by a primary embedding $i : S^2 \hookrightarrow M$ and any two such primary embeddings are isotopic. From Lemma 2.8, the normal bundle of $i : S^2 \hookrightarrow M$ is trivial and we have an embedding $f : S^2 \times D^4 \rightarrow M$ with $f(x, 0) = i$.

Consider the complement $M - f(S^2 \times E^4) \simeq M - i(S^2)$, $\pi_1(M - S^2) = 1$ and $H_*(M - i(S^2), \mathbb{Z}) \cong H_*(S^2, \mathbb{Z})$, which implies $M - i(S^2) \simeq S^2$. Since $\pi_2(M - i(S^2)) \cong \pi_2(M)$, we can represent the generator of $\pi_2(M - i(S^2)) \cong \pi_2(M)$ by a primary embedding $j : S^2 \hookrightarrow M - f(S^2 \times E^4) \subset M$ and $j : S^2 \hookrightarrow M - f(S^2 \times E^4)$ is also a homotopy equivalence by Whitehead theorem. By the standard technique of h -cobordism theorem, we see $M - f(S^2 \times E^4)$ is diffeomorphic to the normal bundle of $j : S^2 \hookrightarrow M$, which is a trivial bundle, and we obtain a diffeomorphism $g : S^2 \times D^4 \cong M - f(S^2 \times E^4)$ with $g(x, 0) = j$.

Finally, we get $M \cong f(S^2 \times D^4) \cup_{f(S^2 \times S^3)} g(S^2 \times D^4) \cong (S^2 \times D^4) \cup_h (S^2 \times D^4)$, where $h = g^{-1}f : S^2 \times S^3 \rightarrow S^2 \times S^3$ is the attaching diffeomorphism. Furthermore, the embeddings f and g are orientation preserving and the two copies of $S^2 \times D^4$ of $(S^2 \times D^4) \cup_h (S^2 \times D^4)$ are also orientation preserving, the attaching map h has to be orientation reversing.

For the degree of $p_2 h j_2$, we see the primary embeddings $i = f(x, 0)$ and $j = g(x, 0)$ are isotopic to each other, which implies $h_* : H_2(S^2 \times S^3, \mathbb{Z}) \rightarrow H_2(S^2 \times S^3, \mathbb{Z})$ is equal to identity. Then on $H_3(S^2 \times S^3, \mathbb{Z})$, $h = -id$ because h is orientation reversing and we get the degree of $p_2 h j_2$ is -1 . By lemma 2.6, the Hopf invariant of $p_1 h j_2$ is d . \square

Proposition 2.11. *If d is odd, any d twisted homology $\mathbb{C}P^3$ is spin.*

Proof. When d is odd, $H^*(M, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[x]/(x^4)$ with $\deg x = 2$ and the homomorphism $Sq^i : H^{6-i}(M, \mathbb{Z}/2\mathbb{Z}) \rightarrow H^6(M, \mathbb{Z}/2\mathbb{Z})$ is zero. Indeed, $Sq^1, Sq^3 = 0$ because $H^{\text{odd}}(M, \mathbb{Z}/2\mathbb{Z}) = 0$ and for the generator $x^2 \in H^4(M, \mathbb{Z}/2\mathbb{Z})$, $Sq^2(x^2) = 2x^3 = 0$. $Sq^* = 0$ implies the total Wu class $v(M)$ of M is equal to 1 and the total Stiefel-Whitney class $w(M) = Sq(v(M)) = 1$. \square

3 Group structure of Π_d^6

3.1 The clutching diffeomorphism f_d

Following [9], let \mathbb{R}^4 be the quaternion field and there exists a differential S^1 action on \mathbb{R}^4 by left multiplication. Under this action, there is a free differential S^1 action on $S^3 \subset \mathbb{R}^4$ whose quotient space S^3/S^1 is just S^2 . For convenience, we identify S^2 by the quotient space $S^3/S^1 := \{S^1x | x \in S^3 \subset \mathbb{R}^4\}$.

We define an orientation-reversing diffeomorphism of $S^2 \times S^3$ by:

$$f : S^2 \times S^3 \rightarrow S^2 \times S^3$$

$$(S^1x, y) \mapsto (S^1xy, y^{-1}),$$

here y^{-1} is the inverse of y in the quaternion field \mathbb{R}^4 . From [9], we know $f \circ f = id$; the degree of the composition of $p_2 \circ f \circ j_2$ is -1 ; and the Hopf invariant of $p_1 \circ f \circ j_2$ is 1.

Let $L : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be a reflection along the hyperplane $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_4 = 0\}$ and $L|_{S^3}$ induces an involution of S^3 with degree -1 , i.e. $L|_{S^3} \circ L|_{S^3} = id$ and $\deg L|_{S^3} = -1$. Define an involution $l : S^2 \times S^3 \rightarrow S^2 \times S^3$ by $l(x, y) = (x, Ly)$.

Then we can define our clutching maps f_d , $d \geq 0$. Let $f_0 := l$, and for $d > 0$, define:

$$f_d := f \circ l \circ f_{d-1}.$$

Since $l \circ l = id$ and $f \circ f = id$, we see $f_1 = f$ and $f_d \circ f_d = id$. Furthermore, by the properties of f and the construction of f_d , these f_d admit special properties:

Lemma 3.1. *f_d is an orientation-reversing diffeomorphism with: (1). $\deg(p_2 \circ f_d \circ j_2) = -1$. (2). The Hopf invariant of $p_1 \circ f_d \circ j_2$ is equal to d .*

Proof. First, l and f are orientation-reversing and $f_d = f \circ l \circ f_{d-1}$, f_d is also orientation reversing.

Second, $H_3(S^2 \times S^3, \mathbb{Z}) = \mathbb{Z}$ and we get $\deg(p_2 \circ f_d \circ j_2) = \deg(p_2 \circ f \circ j_2) \cdot \deg(p_2 \circ l \circ j_2) \cdot \deg(p_2 \circ f_{d-1} \circ j_2) = \deg(p_2 \circ f_{d-1} \circ j_2) = \deg(p_2 \circ f \circ j_2) = -1$.

Finally, $\pi_3(S^2 \times S^3) = \mathbb{Z}a \times \mathbb{Z}b$, where a and b are the generators of $\pi_3(S^2)$ and $\pi_3(S^3)$. For $f : S^2 \times S^3 \rightarrow S^2 \times S^3$, we know $\pi_3(f)(a) = \pi_3(l)(a) = a$, $\pi_3(l)(b) = -b$ and $\pi_3(f)(b) = a - b$. So we obtain $\pi_3(f_d)(b) = da - b$ and the Hopf invariant of $p_1 \circ f_d \circ j_2$ is just equal to d , which is the value of $\pi_3(p_2 \circ f_d)(b) = da$. \square

3.2 The group structure of Π_d^6

Denote Π_d^6 by the set of all diffeomorphism classes of spin d -twisted homology $\mathbb{C}P^3$. We assume $d \geq 0$ because $\Pi_d^6 = \Pi_{-d}^6$.

Let M_1 and M_2 be two elements in Π_d^6 . Since M_1 and M_2 are spin, we can choose two embeddings $h_1 : S^2 \times D^4 \subset M_1$ and $h_2 : S^2 \times D^4 \subset M_2$ such that $h_1(x, 0) : S^2 \hookrightarrow M_1$ and $h_2(x, 0) : S^2 \hookrightarrow M_2$ are primary embeddings of M_1 and M_2 . We define the operation \natural by:

$$M \natural N := (M_1 - h_1(S^2 \times E^4)) \cup_{h_2 f_d h_1^{-1}} (M_2 - h_2(S^2 \times E^4)).$$

Since any primary embeddings of M_i are isotopic as well as their normal bundles, the diffeomorphism class of $M_1 \natural M_2$ is invariant under the variant choices of primary embeddings and their normal bundles. So the operation \natural is well-defined. Furthermore, we have:

Proposition 3.2. $M_1 \natural M_2 \in \Pi_d^6$.

Proof. By proposition 2.10, there exist $k_i : S^2 \times D^4 \subset M_i$, $i = 1, 2$ such that $M_i - h_i(S^2 \times E^4) = k_i(S^2 \times D^4)$. The diffeomorphism $h_i^{-1} k_i : S^2 \times S^3 \rightarrow S^2 \times S^3$ satisfy: (1). the degree of $p_2(h_i^{-1} k_i)j_2$ is -1 , (2) the Hopf invariant of $p_1(h_i^{-1} k_i)j_2$ is equal to d , $i = 1, 2$.

Then $M_1 \natural M_2$ is diffeomorphic to $(S^2 \times D^4) \cup_\lambda (S^2 \times D^4)$, where $\lambda = (k_2^{-1} h_2) f_d (h_1^{-1} k_1)$. We see the diffeomorphism λ satisfies: (1) the degree of $p_2 \lambda j_2$ is -1 , (2). the Hopf invariant of $p_1 \lambda j_2$ is equal to d . By lemma 2.6 and 2.9, $M_1 \natural M_2$ is a spin d -twisted homology $\mathbb{C}P^3$. \square

Theorem 3.3. (Π_d^6, \natural) is an abelian group.

Proof. For any $M \in \Pi_d^6$, by lemma 2.8, $M \cong M(h) = (S^2 \times D^4) \cup_h (S^2 \times D^4)$. We have $M(f_d) \natural M \cong (S^2 \times D^4) \cup_{h f_d f_d} (S^2 \times D^4) = M(h) \cong M$, since $f_d f_d = id$. And we have $M \natural M(f_d) \cong M$ for the same reason. For any $M \cong M(h)$, $M(f_d h^{-1} f_d)$ is also a d -twisted homology $\mathbb{C}P^3$ and $M(h) \natural M(f_d h^{-1} f_d) = M(f_d)$. It is not difficulty to see $M_1 \natural M_2 \cong M_2 \natural M_1$ and \natural is associative. Therefore, (Π_d^6, \natural) is an abelian group with unit $M(f_d)$. \square

Remark 3.4. When $d = 1$, Π_1^6 is just the set of diffeomorphism classes of homotopy $\mathbb{C}P^3$ and this group (Π_1^6, \natural) is just the group defined by Montgomery and Yang in [9] with unit $M(f_1) \cong \mathbb{C}P^3$.

4 The correspondence $\Phi : \Pi_d^6 \rightarrow \Sigma^{6,3}$

4.1 $\Sigma^{6,3}$

Let $\Sigma^{6,3}$ be the set of isotopy classes of embedding of S^3 in S^6 . $\Sigma^{6,3}$ is an abelian group whose operation is defined as follows (cf [2]): let f_1, f_2 be two embedding classes of S^3 in S^6 . Then f_1 and f_2 is isotopic to g_1 and g_2 such that:

(1). $g_1|D_-^3$ is the standard embedding of $D_-^3 \hookrightarrow D_-^6$ and $g_2|D_+^3$ is the standard embedding of $D_+^3 \hookrightarrow D_+^6$. (2). $g_1(D_+^3) \subset D_+^6$, $g_2(D_-^3) \subset D_-^6$. Define $f_1 + f_2$ by:

$$f_1 + f_2(x) = \begin{cases} g_1(x), & x \in D_+^3 \\ g_2(x), & x \in D_-^3 \end{cases}.$$

Under this operation, Haefliger [2] proved that:

Theorem 4.1 (Haefliger). $(\Sigma^{6,3}, +) \cong \mathbb{Z}$.

4.2 $\Phi : \Pi_d^6 \longrightarrow \Sigma^{6,3}$

Now we construct a map from Π_d^6 to $\Sigma^{6,3}$:

First, for any $M \in \Pi_d^6$, choose an arbitrary embedding $h : S^2 \times D^4 \subset M$ such that $h(x, 0)$ is a primary embedding and we also get an embedding $k : S^2 \times D^4 \cong M - h(S^2 \times D^4) \subset M$ with primary embedding $k(x, 0)$. Let $\alpha := f_d h^{-1} k : S^2 \times S^3 \longrightarrow S^2 \times S^3$ be the self diffeomorphism of $S^2 \times S^3$, where f_d was defined in subsection 3.1. We see this clutching map satisfies $\deg(p_2 \alpha j_2) = 1$ and $p_1 \alpha j_2 \simeq pt$. Then one can construct a simply connected closed manifold:

$$P := (M - h(S^2 \times E^4)) \cup_{f_d h^{-1}} (D^3 \times S^3) \cong (S^2 \times D^4) \cup_{\alpha} (D^3 \times S^3)$$

with an embedding $i : S^3 \hookrightarrow D^3 \times S^3 \subset P$. We see P is simply-connected and $H^*(P, \mathbb{Z}) \cong H^*(S^6, \mathbb{Z})$ by Meyer-Vietoris exact sequence. P is diffeomorphic to S^6 because S^6 is the only homotopy sphere in dimension 6. Thus we define:

$$\Phi(M) := [P, i]$$

Since any primary embeddings and their normal bundles are isotopic, Φ is well-defined. On the other hand, Π_d^6 and $\Sigma^{6,3}$ both admit group structures. We can prove:

Proposition 4.2. Φ is a group homomorphism.

Proof. Follow the method of [9], let $M, N \in \Pi_d^6$ and $h : S^2 \times D^4 \subset M$ and $k : S^2 \times D^4 \subset N$ be two embeddings such that $h(x, 0)$ and $k(x, 0)$ are primary embeddings. We know there exist two embeddings $h' : S^2 \times D^4 \cong M - h(S^2 \times E^4)$ and $k' : S^2 \times D^4 \cong N - k(S^2 \times E^4)$ with primary embeddings $h'(x, 0)$ and $k'(x, 0)$. By the definition of Φ , $\Phi(M)$ is the embedding:

$$j_1 : S^3 \hookrightarrow D^3 \times S^3 \subset (M - h(S^2 \times E^4)) \cup_{f_d h^{-1}} (D^3 \times S^3) \cong (S^2 \times D^4) \cup_{\alpha} (D^3 \times S^3),$$

and $\Phi(N)$ is the embedding:

$$j_2 : S^3 \hookrightarrow D^3 \times S^3 \subset (N - k(S^2 \times E^4)) \cup_{f_d k^{-1}} (D^3 \times S^3) \cong (S^2 \times D^4) \cup_{\beta} (D^3 \times S^3),$$

where $\alpha = f_d h^{-1} h'$ and $\beta = f_d k^{-1} k'$ are the self diffeomorphism of $S^2 \times S^3$. We continue to make α and β diffeotopic to the diffeomorphism such that α is identity on $S^2 \times D^3_-$ and β is identity on $S^2 \times D^3_+$. Certainly, $S^2 \times D^4_+ \cup D^3 \times D^3_+ \cong D^6_+$ and $S^2 \times D^4_- \cup D^3 \times D^3_- \cong D^6_-$.

By the definition of the operation \natural : $M \natural N = M - h(S^2 \times E^4) \cup_{k' f_d h^{-1}} N - k'(S^2 \times E^4) \cong h'(S^2 \times D^4) \cup_{k' f_d h^{-1}} k(S^2 \times D^4)$. $\Phi(M \natural N)$ is the embedding:

$$\begin{aligned} j : S^3 \hookrightarrow D^3 \times S^3 &\subset (M \natural N - k(S^2 \times E^4)) \cup_{f_d k^{-1}} D^3 \times S^3 \\ &\cong (S^2 \times D^4) \cup_{f_d k^{-1} k' f_d h^{-1} h'} (D^3 \times S^3) \cong (S^2 \times D^4) \cup_{\beta \alpha} (D^3 \times S^3). \end{aligned}$$

We see j is just the sum of j_2 and j_1 under $\Sigma^{6,3}$ and we obtain $\Phi(M \natural N) = \Phi(M) + \Phi(N)$. \square

Theorem 4.3. $\Phi : \Pi_d^6 \longrightarrow \Sigma^{6,3}$ is isomorphic.

Proof. For any embedding $j : S^3 \hookrightarrow S^6$, the normal bundle of j is trivial and there exists an embedding $k : S^3 \times D^3 \subset S^6$ with $k(x, 0) = j$. The complement $S^6 - k(S^3 \times E^3)$ is homotopy equivalent to S^2 and by h -cobordism theorem, we have a diffeomorphism $h : S^2 \times D^4 \cong S^6 - k(S^3 \times E^3)$. So we have a decomposition $S^6 = h(S^2 \times D^4) \cup k(D^3 \times S^3) \cong S^2 \times D^4 \cup_{kh^{-1}} D^3 \times S^3$, where kh^{-1} is the self diffeomorphism of $S^2 \times S^3$. By the technique of [9] (cf p494), one can modify k to make the attaching map kh^{-1} admit $\deg p_2(kh^{-1})j_2 = 1$ and $p_1(kh^{-1})j_2 \simeq pt$.

Define $M(\psi) := (S^2 \times D^4) \cup_\psi (S^2 \times D^4)$, where $\psi = f_d(kh^{-1})$. Then we see $M(\psi) \in \Pi_d^6$ and $\Phi(M) = [S^6, j] \in \Sigma^{6,3}$. So Φ is surjective. Next, we want to prove Φ is injective.

For $M_1 = (S^2 \times D^4) \cup_{\lambda_1} (S^2 \times D^4)$ and $M_2 = (S^2 \times D^4) \cup_{\lambda_2} (S^2 \times D^4)$, if the embeddings $\Phi(M_1) = j_1 : S^3 \hookrightarrow D^3 \times S^3 \subset (S^2 \times D^4) \cup_{f_d \lambda_1} (D^3 \times S^3)$ and $\Phi(M_2) = j_2 : S^3 \hookrightarrow D^3 \times S^3 \subset (S^2 \times D^4) \cup_{f_d \lambda_2} (D^3 \times S^3)$ are isotopic. Then by the tubular neighborhood theorem of j_1, j_2 , after an isotopy which does not affect the diffeomorphism type, there exists a diffeomorphism $G : S^2 \times S^3 \longrightarrow S^2 \times S^3$, $G(x, y) = (g(y)x, y)$ such that $\lambda_1 = \lambda_2 G$, where $g : S^3 \longrightarrow SO(3)$. Since the Hopf invariant of $p_1 \lambda_1 j_2$ and $p_1 \lambda_2 j_2$ are the same, which implies $p_2 G j_2 = p_2 g = 0 \in \pi_3(S^2)$. Since $\pi_3(SO(3)) \cong \pi_3(SO(3)/SO(2)) = \pi_3(S^2)$, we see $g \simeq pt$ and λ_1 is isotopic to λ_2 and $M_1 \cong M_2$. \square

Remark 4.4. From Wall's paper [10], we see for $M \in \Pi_d^6$ with $H^*(M, \mathbb{Z}) = \mathbb{Z}[x, y]/(x^2 - dy, y^2)$, $\deg x = 2$, $\deg y = 4$, the first Pontrjagin class $p_1(M)$ of M is equal to $(4d - 24\Phi(M))y$. If $\Phi(M_1) = \Phi(M_2)$, by the classification theorem of simply connected closed spin 6-manifolds with torsion free integral homology (cf [10]), $M_1 \cong M_2$.

4.3 Homotopy type of Π_d^6

When $d = 1$, Π_1^6 is the set of diffeomorphism classes of homotopy $\mathbb{C}P^3$, what about the set Π_d^6 , $d > 1$? In their paper [4], Libgober and Wood proved:

Theorem 4.5 (Libgober, Wood). *If $n = 2m + 1$ and if d has not divisors less than $m + 2$, then any two d -twisted homology $\mathbb{C}P^3$ are homotopy equivalent.*

When $n = 3$ and d is odd, we get:

Corollary 4.6. *If d is odd, for any $M_1, M_2 \in \Pi_d^6$, $M_1 \simeq M_2$.*

When d is even and for $M_1, M_2 \in \Pi_d^6$, from [4] section 9 and [10] section 7 we know $M_1 \simeq M_2$ if and only if their first Pontrjagin classes satisfy $p_1(M_1) \equiv p_1(M_2) \pmod{48}$, where $p_1(M_i) \in H^4(M_i, \mathbb{Z}) = \mathbb{Z}$, $i = 1, 2$. Furthermore, $p_1(M_i) = 4d - 24\Phi(M_i)$, where $\Phi : \Pi_d^6 \longrightarrow \Sigma^{3,3} \cong \mathbb{Z}$ is the isomorphism constructed above (cf [10] section 6). We obtain:

Proposition 4.7. *If d is even, for any $M_1, M_2 \in \Phi^{-1}(2\mathbb{Z})$ or $\Phi^{-1}(2\mathbb{Z} + 1)$, $M_1 \simeq M_2$.*

5 Application to the free involution of d -twisted homology $\mathbb{C}P^3$

In [5], Bang-he Li and Zhi Lü proved

Theorem 5.1 (Li-Lü). *For any embedding $i : S^3 \hookrightarrow S^6$, there exists an involution σ on S^6 such that the fixed point is $i(S^3)$.*

In this section, we will use their result and the correspondence Φ constructed above to prove:

Theorem 5.2. *For any $M \in \Pi_d^6$, there exists a free involution on M .*

Proof. For any embedding $i : S^3 \hookrightarrow S^6$, choose such an involution $\sigma : S^6 \rightarrow S^6$ with fixed point $i(S^3)$. By the equivariant tubular neighborhood theorem, there exists a $\mathbb{Z}/2\mathbb{Z}$ equivariant embedding $h : S^3 \times D^3 \rightarrow S^6$ with $\sigma h(x, y) = h(x, -y)$, where the action of $\mathbb{Z}/2\mathbb{Z} \curvearrowright S^3 \times D^3$ is $(x, y) \mapsto (x, -y)$. On the other hand, σ is free on $S^6 - h(S^3 \times D^3) \cong S^2 \times D^4$. So on $S^6 = (S^2 \times D^4) \cup_\lambda (S^3 \times D^3)$, the involution restricted on the part of $S^3 \times D^3$ can be $\sigma|_{S^3 \times D^3} : (x, y) \mapsto (x, -y)$. By the technique of [9], we can make the diffeomorphism λ satisfy $\deg(p_2 \lambda j_2) = 1$ and $p_1 \lambda j_2 \simeq pt$.

Define a free involution ι on $S^2 \times D^4$ by $\iota(x, y) = (-x, y)$. We see $\iota \lambda = \lambda \sigma$ and $\iota f_d = f_d \iota$ on $S^2 \times S^3$. For the manifold, $(S_a^2 \times D^4) \cup_{f_d \lambda} (S_b^2 \times D^4)$, where S_a^2, S_b^2 are 2-spheres with label a and b , we see $\Phi((S_a^2 \times D^4) \cup_{f_d \lambda} (S_b^2 \times D^4)) = i : S^3 \hookrightarrow S^6$. We define a free involution δ on $(S_a^2 \times D^4) \cup_{f_d \lambda} (S_b^2 \times D^4)$ by $\delta|_{S_a^2 \times D^4} = \sigma$ and $\delta|_{S_b^2 \times D^4} = \iota$. On the boundary $S^2 \times S^3$ of these two copies, we see $\delta f_d \lambda = \iota f_d \lambda = f_d \iota \lambda = f_d \lambda \sigma = f_d \lambda \delta$. So δ is well-defined and is also a free involution.

In section 4, we've proved that Φ is an isomorphism. Thus, we conclude that for any $M \in \Pi_d^6$, there always exists a free involution. \square

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